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DYNAMIC ANALYSIS OF ELASTIC-PLASTIC BEAMS BY MEANS OF THERMOELASTIC SOLUTIONS†

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Abstract—The equation of bending motion of the inelastic beam is put into a linear form by interpreting the nonlinear part of the operator as a fictitious temperature-moment loading, thus consistently considering the inhomogeneous character of the dynamic boundary conditions. The total solution is separated into two parts: a quasistatic deflection due to external loadings and fictitious temperature moments and a dynamic part of the deflection due to inertial forces and homogeneous boundary conditions. The numerical procedure is of an iterative and time-step type and is equivalent to the Newton-Raphson method in matrix formulation.

1. INTRODUCTION

Well-known procedures make use of an incremental formulation, thereby altering the stiffness of the structure which is considered elastic in each incremental step. In practice, there is a need for keeping the stiffness matrix constant during the computation of the elastic-plastic response.

Following an idea of Timoshenko and Goodier[1], who considered thermoelastic disturbances equivalent to distributed body-forces and surface tractions acting on an isothermal elastic body, which was extended to elastic-plastic behavior with respect to a fictitious loading of the purely elastic body in [2], an alternative procedure is presented and applied to a beam under various thermal and isothermal transient loadings. Similarly, inelastic plates and shells may be considered. The equation of bending motion of the inelastic beam is put into a linear form by interpreting the plastic part of the operator as a fictitious *temperature-moment* load, thus consistently considering the inhomogeneous character of the dynamic boundary conditions, and leaving the kinematical boundary conditions unaltered. Next we separate the total solution into two parts: (1) The quasistatic deflection due to external loadings and fictitious temperature moments. Computation is made by standard linear elastic analysis, e.g. using Greens' function and Maysels' formula of thermoelasticity. These solutions eventually take care of inhomogeneous dynamic boundary conditions; (2) The dynamic part of the deflection due to inertial loadings and homogeneous boundary conditions. In that part, we apply Galerkin's procedure, using elastic eigenfunctions for the Ritz-Ansatz, rendering an uncoupled set of one-degree-of-freedom linear elastic oscillators in forced vibrations. The forcing functions have to be calculated using the quasistatic solution. The numerical procedure is of an iterative and time-step type which is adjusted to the above formulation. The first step of calculation considers external loadings only, thus rendering the elastic solution for all times.

In a second step, time is discretized and the fictitious temperature moment is calculated from total strain assuming a proper elastic-plastic materials law. The procedure is equivalent to the Newton-Raphson method. In the course of the calculations, the elastic part of the solution is unaltered and the given loadings do not enter the iterative procedure. Because only linear operators with constant coefficients determine the formulation, it is believed that the method will be of interest in the course of nonlinear engineering dynamics of elastic-plastic bodies. In addition, due to the modal analysis no solutions of systems of equations are required.

† The authors expressively dedicate this paper in memoriam to Professor Alicia Golebiewska Herrmann, late Associate Editor of the *International Journal of Solids and Structures*.

2. BASIC EQUATIONS

The equation of motion of a beam is

$$M_{,xx} = -q + \rho A \ddot{w}. \tag{2.1}$$

In simple Bernoulli-Euler theory, where rotational inertia is neglected and where linear distribution of strain is assumed in the form $\epsilon = -zw_{,xx}$, the constitutive equation may be reduced to

$$M = EJ(\kappa - m - m_\theta), \qquad \kappa = -w_{,xx}. \tag{2.2}$$

The external distributed loading is denoted $q(x, t)$ and

$$m_\theta = \frac{1}{J}\int_A \alpha\theta z \, dA \tag{2.3}$$

represents an external temperature loading $\theta(x, z, t)$, which is considered in [3] and omitted in this study. A is the cross-sectional area of the beam, I is moment of inertia, E denotes the initial Young's modulus, w is the deflection and M is the bending moment. Equation (2.2) follows from the nonlinear stress-strain relationship

$$\sigma = E(\epsilon - \epsilon^{(N)} - \alpha\theta) \tag{2.4}$$

which, when integrated, renders

$$m = \frac{1}{J}\int_A z\epsilon^{(N)} \, dA. \tag{2.5}$$

This expression is analogous to eqn (2.3), noting that a nonlinear temperature distribution, $\alpha\theta$, corresponds to the nonlinear strain $\epsilon^{(N)}$. Elimination of M leads to the integro-differential-equation, we assume constant initial bending stiffness,

$$EJ(-\kappa_{,xx} + m_{,xx}) + \rho A \ddot{w} = q. \tag{2.6}$$

The set of boundary conditions to eqn (2.6) consists of two kinematical conditions,

$$KBC: \quad w = 0, \qquad w_{,x} = 0 \tag{2.7}$$

and the inhomogeneous dynamical conditions,

$$DBC: \quad -\kappa + m = 0, \quad (M = 0): \quad -\kappa_{,x} + m_{,x} = 0, \quad (Q = M_{,x} = 0). \tag{2.8}$$

Note the analogy to linear thermal bending problems[4, p. 220], which renders the same type of inhomogeneous DBC. Similarly, the dynamical conditions of continuity of nonlinear deflection in the case of a single force load and in the case of a singular external moment loading are in full analogy to a linear beam with temperature loading. Hence, we apply the methods of solution of linear thermal-shock problems, see e.g. [5, p. 108] and [6], to solve the nonlinear isothermal vibration problem.

3. SOLUTION IN ANALOGY TO LINEAR THERMAL SHOCK LOADING

The key step in the solution procedure is the splitting of the deflection w in two parts, a quasistatic deformation w^S with inhomogeneous boundary conditions (2.8) and a dynamic deflection w^D applying homogeneous DBC:

$$w = w^S + w^D, \tag{3.1}$$

$$EJ(-\kappa^S{}_{,xx} + m_{,xx}) = q,$$
$$KBC: \quad w^S = 0, \quad w^S{}_{,x} = 0, \tag{3.2}$$
$$DBC: \quad \kappa^S = m, \quad \kappa^S{}_{,x} = m_{,x}$$

$$-EJ\kappa^D{}_{,xx} + \rho A\ddot{w}^D = \hat{q}, \quad \hat{q} = -\rho A\ddot{w}^S,$$
$$\text{KBC: } w^D = 0, \quad w^D{}_{,x} = 0, \tag{3.3}$$
$$\text{DBC: } \kappa^D = 0, \quad \kappa^D{}_{,x} = 0.$$

Both parts of the solution are expressed by the linear elastic deflection, w^*, superimposed by the nonlinear deformation, w^{**}.

3.1 *Quasistatic deflection*

Putting $m \equiv 0$ in eqn (3.2) and solving the linear elastostatic equations by any convenient standard procedure renders w^{*S} at any instant of time. Assuming the fictitious "thermal" loading m of the linear beam to be known, the remaining nonlinear deformation w^{**S} can be calculated by powerful methods of solution of linear thermoelasticity (e.g. by Mohr's analogy or Maysel's formula, etc.), and

$$w^S = w^{*S} + w^{**S}. \tag{3.4}$$

The simple case of a hinged-hinged beam of span l under constant loading q, will be considered here as an example:

$$w^{*S} = \frac{ql^4}{24EJ}(\xi - 2\xi^3 + \xi^4), \quad M^{*S} = \frac{ql^2}{2}(\xi - \xi^2), \quad \xi = x/l \tag{3.5}$$

and, because the support is statically determinate, $M^{**S} = 0$, and hence integrating eqn (2.2) by Mohr's analogy renders

$$w^{**S} = l^2\left[\int_0^{\xi} \bar{\xi} m \, d\bar{\xi} + \xi \int_{\xi}^{1/2} m \, d\bar{\xi}\right]. \tag{3.6}$$

3.2 *Dynamic deflection*

Inserting $w^{**S} = 0$ and solving eqn (3.3) renders the linear part w^{*D} of the dynamic deflection. Homogeneous b.c. allow for the application of the classical Ritz-Galerkin procedure,

$$w^{*D} = \sum_{i=1}^{N} q_i^*(t)\varphi_i(\xi), \quad \xi = x/l. \tag{3.7}$$

When $\varphi_i(\xi)$ are the orthogonal eigenfunctions of the linear beam with homogeneous DBC, $\varphi_i \equiv W_i(\xi)$, the time functions $q_i^*(t)$ are determined by the uncoupled solutions of n oscillator equations:

$$\ddot{q}_i^* + \omega_i^2 q_i^* = F_i^*, \quad i = 1 \ldots N, \tag{3.8}$$

when ω_i are the eigenfrequencies of the linear elastic beam with homogeneous DBC

and without plastic deformations and the generalized force is given through the integral

$$F_i^* = -\frac{\rho A l}{m_i}\int_0^1 w^{*S} W_i \, d\xi, \qquad m_i = \rho A l \int_0^1 W_i^2 \, d\xi. \tag{3.9}$$

For the hinged-hinged beam:

$$W_k = \sin k\pi\xi, \qquad \omega_k = k^2\pi^2\sqrt{EJ/\rho A l^4} \tag{3.10}$$

and

$$q_k^* = \frac{-4l^4}{(\pi k)^5 EJ}\left[q_0 \cos \omega_k t + \frac{\nu\omega_k q_1}{(\omega_k^2 - \nu^2)}\left(\sin \omega_k t - \frac{\nu}{\omega_k}\sin \nu t\right)\right],$$
$$k = 2n - 1, \quad n = 1, 2, 3 \ldots, \tag{3.11}$$

if the loading is assumed independent of x in the form

$$q = q_0 + q_1 \sin \nu t, \qquad t \geq 0. \tag{3.12}$$

The dynamic deflection w^{**D} to be superimposed on the linear part is decomposed analogous to eqn (3.7) and the time-functions q_i^{**} are the solutions of the oscillator equations, $\varphi_i = W_i$,

$$\ddot{q}_i^{**} + \omega_i^2 q_i^{**} = F_i^{**}, \qquad i = 1 \ldots N, \tag{3.13}$$

with new forcing functions F_i^{**} depending on the nonlinear quasistatic deflection w^{**S},

$$F_i^{**} = -\frac{\rho A l}{m_i}\int_0^1 \ddot{w}^{**S} W_i \, d\xi. \tag{3.14}$$

Equation (3.7), in more general cases, can be replaced by the *FEM*-approximation and the sets of eqns (3.8), (3.13), respectively, may become coupled, because the stiffness matrix is, in general, nondiagonal. With $N \to \infty$ the solutions given so far become exact. Approximation enters only by evaluating the nonlinear "temperature" loading for a nonlinear stress-strain or, in this study, nonlinear bending moment-curvature relation in discrete points of the beam only.

The solution to eqns (3.8), (3.13) in time-domain is by convolution, it can be conveniently given in frequency domain first and afterwards transformed to time domain by

$$q_k|^{*,**} = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_k(\bar{\omega}) C_k^{*,**}(\bar{\omega}) e^{i\bar{\omega}t} \, d\bar{\omega}, \tag{3.15}$$

where, cf.[7],

$$H_k(\bar{\omega}) = 1/m_k(\omega_k^2 - \bar{\omega}^2), \tag{3.16}$$

and

$$C_k^{*,**} = \int_{-\infty}^{\infty} F_k^{*,**}(t) e^{-i\bar{\omega}t} \, dt. \tag{3.17}$$

can be evaluated by the Fast Fourier Transform (*FFT*).

3.3 *Assumptions on the nonlinear loading m(x, t)*

We assume an external loading $q(x, t)$ which separates in space-time variables.

Hence, considering p_t incremental time-steps within observation time t,

$$m(x, t) = \sum_{j=1}^{p_t} \Delta m_j(x).f(t - j\Delta t). \tag{3.18}$$

Conveniently,

$$f(t - j\,\Delta t) = H(t - j\,\Delta t) \quad \text{or} \quad f(t - j\,\Delta t) = [r(t - t_{j-1}) - r(t - t_j)]/\Delta t, \tag{3.19}$$

where $H(t)$ is the Heaviside step function and $r(t)$ is a linear ramp function.

We use a hinged-hinged beam in quasistatic and pure bending motion for illustration:

$$M^{*S}(x, t) = M_0\lambda(t). \tag{3.20}$$

Assuming the state of the nonlinear beam known at time t_p, the linear part of the solution changes at t_{p+1} by the amount

$$\Delta\kappa_{p+1}^{*S} = \frac{M_0}{EJ}[\lambda(t_{p+1}) - \lambda(t_p)]. \tag{3.21}$$

In this case, Δm_j = const. over the span of the beam. Note the interpretation of nonlinear strains as fictitious loadings as proposed in [2] fails without additional considerations of b.c. In the present formulation, however, material nonlinearity renders simply

$$\kappa_{p+1}^{**S} = \sum_{j=1}^{p+1} \Delta m_j, \qquad \Delta\kappa_{p+1}^{**S} = \Delta m_{p+1}. \tag{3.22}$$

Hence, dropping S

$$\Delta\kappa_{p+1} = \Delta\kappa_{p+1}^{*} + \Delta m_{p+1} \tag{3.23}$$

is a nonlinear equation for the curvature increment $\Delta\kappa_{p+1}$, since $\Delta m(\Delta\kappa)$ may be taken from the generalized constitutive law $M(\kappa)$, see Fig. 1, cf. eqn (2.2).

Equation (3.23) is accompanied by the constraint (unloading),

$$\Delta m_{p+1} = 0, \tag{3.24}$$

whenever $M(\kappa)$ is in the linear paths of Fig. 1. An iterative solution in the form

$$\Delta\kappa^{(i)} = \Delta m(\Delta\kappa^{(i-1)}) + \Delta\kappa^*, \quad i = 1, 2, \ldots, \tag{3.25}$$

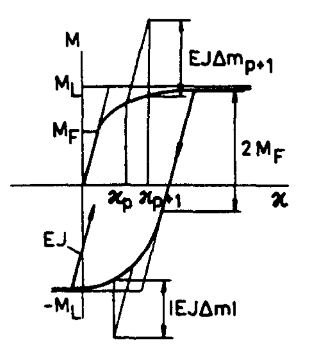

Fig. 1. $M - \kappa$ constitutive relation of ideal elastic-plastic material for a beam with symmetric cross-section. Definition of fictitious temperature moment increment Δm.

with a start value of $\Delta m(\Delta\kappa^{(0)}) = 0$—the time indicating index has been dropped—is exactly the modified Newton-Raphson method, [8, p. 291], [9, p. 454]: The roots of the equation

$$y(x) = f(x) + c = 0 \tag{3.26}$$

are iteratively determined from

$$x^{(i)} = x^{(i-1)} - [f(x^{(i-1)}) + c]/f'(x^{(i-1)}). \tag{3.27}$$

In the above case we identify $x \equiv \Delta\kappa$, $f = \Delta m - \Delta\kappa$, $c = \Delta\kappa^*$ and put $f'(x^{(0)}) = -1$. Convergence of the iteration is assured in the neighbourhood of the root. The speed of convergence is determined by the time interval Δt, i.e. by the value of the linear elastic curvature increment $\Delta\kappa^*$. Close to points with horizontal tangents in the $M(\kappa)$ graph, an updated version of the Newton-Raphson method gives an accelerated convergence: f' is chosen at the point of decelerating convergence and then kept constant:

$$\Delta\kappa^{(i)} = \Delta\kappa^{(i-1)} - [\Delta m(\Delta\kappa^{(i-1)}) - \Delta\kappa^{(i-1)} + \Delta\kappa^*]/[\Delta m'(\Delta\kappa^{(h)}) - 1],\ h < i. \tag{3.28}$$

4. ITERATIVE SOLUTION OF THE NONLINEAR EQUATIONS

In the case of dynamic loading $q(x, t)$ we set $\Delta m_j(x)$ of eqn (3.18) constant in each of the intervals Δx_k, $k = 1 \ldots p_1$ of the span, $l = p_1 \Delta x$:

$$\Delta m_j(x) = \sum_{k=1}^{p_1} \Delta m_{jk}[H(x - x_{j-1}) - H(x - x_j)]. \tag{4.1}$$

Hence, from eqn (2.2), with $M^{**S} = 0$ in statically determinate beams, Fig. 2,

$$\Delta\kappa_{jk}^{**S} = \Delta m_{jk} \tag{4.2}$$

in the kth interval, and zero otherwise. Using ramp functions in eqn (3.18) and the eigenfunctions of the hinged-hinged beam in eqn (3.14), the dynamic forcing functions become

$$F_{ijk}^{**} = f_{ik}^{**}[\delta(t - t_{j-1}) - \delta(t - t_j)]\, \Delta m_{jk}/\Delta t \tag{4.3}$$

where

$$f_{ik}^{**} = -\frac{2\rho A l^3 \Delta\xi_k}{(i\pi)^2 m_i} \sin i\pi\xi_k,\quad i = 1, 3, 5, \ldots, \text{and zero otherwise.} \tag{4.4}$$

Above, the deflection Δw_{jk}^{**S} of Fig. 2 has been used.

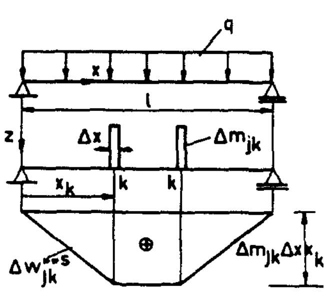

Fig. 2. Example beam of span l and loading $q(t)$. Quasistatic deflection increment due to loading Δm_{jk}, $j = p + 1$ in Fig. 1.

The solution of eqn (3.13) becomes simply

$$q_{ijk}^{**}(t) = \frac{f_{ik}^{**}}{\omega_i \Delta t}\,[H(t - t_{j-1}) \sin \omega_i(t - t_{j-1}) - H(t - t_j) \sin \omega_i(t - t_j)]\, \Delta m_{jk}. \quad (4.5)$$

Superposition of the static curvature, eqn (3.22), and of the dynamic part renders

$$\Delta \kappa_{pr}^{**} = \Delta m_{pr} + \Delta \kappa_{pr}^{**D}, \quad (4.6)$$

where

$$\Delta \kappa_{pr}^{**D} = \Delta \kappa_{pr}^{**D'} + \Delta \kappa_{pr}^{**D''}, \quad (4.7)$$

and using the incremental form of eqns (3.7) and (2.2),

$$\Delta \kappa_{pr}^{**D'} = \sum_{j=1}^{p-1} \sum_{k=1}^{p_1} \sum_{i=1}^{N} q_{ijk}^{**}(t_p) \left(\frac{i\pi}{l}\right)^2 W_i(\xi_r), \quad (4.8)$$

is the curvature increment due to the Δm's already calculated, and

$$\Delta \kappa_{pr}^{**D''} = \sum_{k=1}^{p_1} \sum_{i=1}^{N} q_{ipk}^{**}(t_p) \left(\frac{i\pi}{l}\right)^2 W_i(\xi_r) \quad (4.9)$$

contains the unknown spanwise distributed Δm_{pk} to be evaluated at time $t = t_p = p\Delta t$. For a numerical evaluation of eqn (4.8) see Appendix A.

Further superposition renders

$$\Delta \kappa_{pr} = \Delta \kappa_{pr}^{*} + \Delta \kappa_{pr}^{**}, \quad (4.10)$$

which is the nonlinear system of equations for the p_1 increments of curvature $\Delta \kappa_{pr}$ with $r = 1 \ldots p_1$, due to Δm_{pr}. Equation (4.10) is the matrix equivalent to eqn (3.23) discussed before:

$$\Delta \kappa_p = \Delta \kappa_p^{*} + \Delta \kappa_p^{**D'} + \mathbf{G}_p^{**}\, \Delta \mathbf{m}_p, \quad (4.11)$$

where

$$\Delta \kappa_p = [\Delta \kappa_{p1} \ldots \Delta \kappa_{pr} \ldots \Delta \kappa_{pp_1}]^T, \quad (4.12)$$

$$\Delta \kappa_p^{*} = [\Delta \kappa_{p1}^{*} \ldots \Delta \kappa_{pr}^{*} \ldots \Delta \kappa_{pp_1}^{*}]^T, \quad (4.13)$$

$$\Delta \kappa_p^{**D'} = [\Delta \kappa_{p1}^{**D'} \ldots \Delta \kappa_{pr}^{**D'} \ldots \Delta \kappa_{pp_1}^{**D'}]^T,$$

$$\Delta \mathbf{m}_p = [\Delta m_{p1} \ldots \Delta m_{pr} \ldots \Delta m_{pp_1}]^T, \quad (4.14)$$

and

$$\mathbf{G}_p^{**} = \begin{pmatrix} g_{p11}^{**} + 1 & \ldots & g_{p1p_1}^{**} \\ \vdots & g_{prr}^{**} + 1 & \vdots \\ g_{pp_11}^{**} & \ldots & g_{pp_1p_1} + 1 \end{pmatrix} \quad (4.15)$$

have the elements

$$g_{prk}^{**} = \sum_{i=1,2}^{N} \frac{f_{(2i-1)k}^{**}}{\omega_{2i-1}\, \Delta t}\, \frac{\pi^2(2i - 1)^2}{l^2}\, W_{2i-1}(\xi_r) \sin \omega_{2i-1}(t_p - t_{p-1})$$

$$r = 1 \ldots p_1, \quad k = 1 \ldots p_1. \quad (4.16)$$

A modified Newton-Raphson procedure can be applied and we iterate

$$\Delta\kappa_p^{(i)} = \Delta\kappa_p^{*} + \Delta\kappa_p^{**D'} + \mathbf{G}_p^{**}\,\Delta\mathbf{m}_p(\Delta\kappa_p^{(i-1)}) \tag{4.17}$$

with start vector $\Delta\mathbf{m}_p(\Delta\kappa^{(0)}) = \mathbf{0}$ and the components of $\Delta\mathbf{m}_p$ being zero for linear parts of the $M(\kappa)$ constitutive law. A more general procedure is outlined in Appendix B. Updating renders

$$\Delta\kappa_p^{(i)} = \Delta\kappa_p^{i-1} + [\Delta\kappa_p^{(i-1)} - \mathbf{G}_p^{**}\,\Delta\mathbf{m}_p(\Delta\kappa^{(i-1)}) - \Delta\kappa_p^{*} - \Delta\kappa_p^{**D'}]\mathbf{K}^{-1}, \tag{4.18}$$

where

$$\mathbf{K} = \begin{pmatrix} (g_{p11}^{**} + 1)\Delta m_{p1}'(\Delta\kappa_{p1}^{(k)}) - 1 & 0 \\ 0 & (g_{ppipi}^{**} + 1)\Delta m_{ppi}'(\Delta\kappa_{ppi}^{(k)}) - 1 \end{pmatrix}. \tag{4.19}$$

Convergence properties are same as in the scalar case,[9].

Beams with statically indeterminate b.c., hence, with nonvanishing bending moment M^{**S}, have nonvanishing static curvature increments also for noncoinciding span elements, and $\mathbf{G}_p^{**}$ contains additional terms.

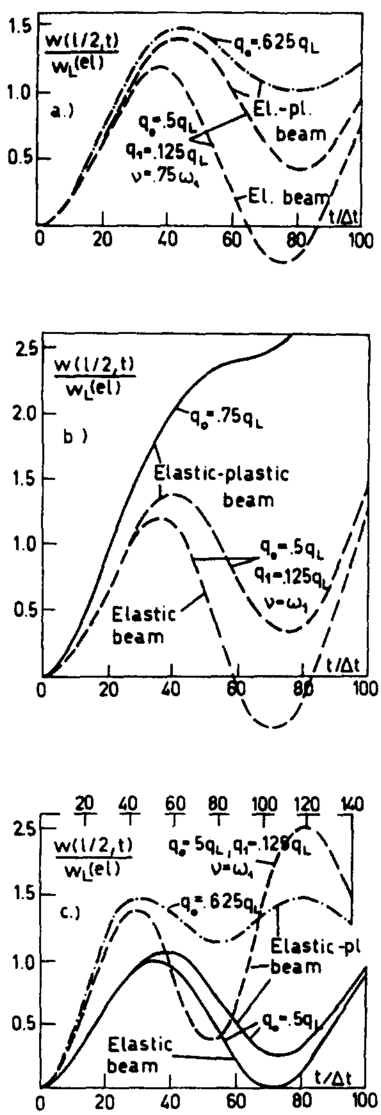

Fig. 3(a, b, c). Dimensionless midspan deflection versus dimensionless time. Δt . . . increment, $w_L^{(elast)} = 5q_L l^4/384EJ$.

5. NUMERICAL RESULTS

For comparison's sake, cf.[2], we consider a hinged-hinged beam with rectangular cross-section $B \times H$, $H/B = 2$, and $l/B = 30$. Material properties are $E/\sigma_F = 600$, where σ_F is the yield stress of an ideal elastic-plastic material and $\sqrt{EJ/\rho A l^4} = 129.78[s^{-1}]$. The $M - \kappa$ constitutive relation for that case is given in Fig. 1, and the nonlinear loading path becomes

$$M = \tfrac{3}{2} M_F \left(1 - \frac{\kappa_F^2}{3\kappa^2}\right), \tag{5.1}$$

where $\kappa_F = M_F/EJ$.

First, we consider a step lateral load q_0 with $0.5q_L$, $0.625q_L$, and $0.75q_L$, respectively, where $q_L = 12M_F/l^2$ is the static collapse load which causes a plastic hinge at the midspan of the beam. Secondly, a sinusoidal load is superposed where $q_1 = .125q_L$ is the amplitude and $\nu = 2\omega_1/3$ and ω_1 are the values of the forcing frequency to be considered. The "resonant" case $\nu = \omega_1$ requires a reformulation of eqn (3.11) to

$$q_1^* = \frac{-4l^4}{\pi^5 EJ}\left[q_0 \cos \omega_1 t + \frac{q_1}{2}(\sin \omega_1 t + \omega_1 t \cos \omega_1 t)\right], \tag{5.2}$$

since no viscous damping is considered here.

Results are cast into graphical form in Fig. 3 with the observation time $100 \times \Delta t$. Time-increments are same as those of [2], $\Delta t = 2\pi/72\omega_1 = 0.681 \times 10^{-4}[s]$ for comparison's sake, also, spanwise, the number of intervals is chosen to be 15, likewise to [2], where only step-loading was considered. In contrast to [2], no discretization within the cross-section is used. However, the influence of the shearing force is neglected.

For higher loadings, the decrease of deflections in Fig. 3 is about 30% less compared to [2] for $q_0 = 0.625q_L$ and is absent for $q_0 = 0.75q_L$, where repeated plastification during unloading occurs.

6. CONCLUSION

Interpretation of the nonlinar elasto-plastic problem in analogy to linear thermoelasticity renders an efficient algorithm which, for a realistic $M - \kappa$ constitutive relation, is unconditionally stable and may be interpreted as a modified Newton-Raphson method. Test calculations showed insensitivity to the choice of the value of Δt. Reduction of mesh size within the span does not considerably increase computer costs.

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APPENDIX A

Equation (4.8) determines the curvature increment in the pth time interval due to the previously calculated Δm's. This form, however, has to be updated in order to keep the computational effort equally low for all p. Hence, summation over j should be avoided, which is enabled by the algorithm given below.

Consider the jth time interval. That part of the ith generalized coordinate, which is due to Δm_j, may be written as (see eqn (4.5)),

$$q_{ij}^{**} = \sum_{k=1}^{pl} q_{ijk}^{***}. \tag{A.1}$$

(Note that $q_{ij}^{**} = 0$, $t < t_{j-1}$). The remaining part of q_i^{**} conveniently may be calculated from the initial values at the time t_{j-1}. Evaluating q_i^{**} at the time nodes j, a storage algorithm is thus rendered:

$$q_i^{**}(t_j) = q_{ij}^{**}(t_j) + q_i^{**}(t_{j-1}) \cos \omega_i \Delta t + \frac{\dot{q}_i^{**}(t_{j-1})}{\omega_i} \sin \omega_i \Delta t. \tag{A.2}$$

Furthermore,

$$\dot{q}_i^{**}(t_j) = \dot{q}_{ij}^{**}(t_j) - q_i^{**}(t_{j-1})\omega_i \sin \omega_i \Delta t + \dot{q}_i^{**}(t_{j-1}) \cos \omega_i \Delta t. \tag{A.3}$$

Note that $\dot{q}_{ij}^{**}$ in eqn (A.3) has to be evaluated at $t_i + 0$.

Inserting (A.2) and (A.3) into eqn (4.8) gives the desired form of proper incrementation:

$$\Delta \kappa_{pr}^{**D'} = \sum_{i=1}^{n} \left[q_i^{**}(t_{p-1}) \cos \omega_i \Delta t + \frac{\dot{q}_i^{**}(t_{p-1})}{\omega_i} \sin \omega_i \Delta t \right] \left(\frac{i\pi}{l} \right)^2 W_i(\xi_r). \tag{A.4}$$

APPENDIX B

The formulation developed so far makes use of moment-curvature relations, which seems to be a straightforward strategy as long as the cross-section of the beam is symmetric and the proper $M(\kappa)$-diagram may be conveniently derived, see e.g. eqn (5.1), or taken from the literature directly.

In case of (plane bending of) beams with non-symmetrical cross-sections, however, the neutral axis changes, if yielding takes place, see e.g. [10]. Furthermore, use of more complicated cross-sectional areas shows the need of numerical procedures, which are free from analytical $M(\kappa)$-relations. Such a method may easily be implemented and adopted for the solution of the nonlinear system of equations (4.11), which is shown below.

The cross-sectional area is discretized into p_A stripes of area ΔA_s, $s = 1, \ldots, p_A$, where $A = \sum_{s=1}^{p_A} \Delta A_s$. The increment of strain in the stripe of number s is found from the linear distribution

$$\Delta \epsilon_{prs} = z_s \Delta \kappa_{pr} + \Delta \epsilon_{pr}^{(0)}, \tag{B.1}$$

where $\epsilon^{(0)} = \epsilon(z = 0)$ and z is measured from the cross section's center of gravity. Equilibrium requires

$$N = \int_A \sigma \, dA = EA(\epsilon^{(0)} - n) = 0, \tag{B.2}$$

where

$$n = \frac{1}{A} \int_A \epsilon^{(N)} \, dA \tag{B.3}$$

denotes the fictitious mean temperature of the beam. Hence, $\Delta\epsilon^{(0)} = \Delta n$ and

$$\Delta \epsilon_{prs} = z_s \Delta \kappa_{pr} + \Delta n_{pr}. \tag{B.4}$$

For the iterative solution of eqn (4.11), a first approximation is calculated using $\Delta \mathbf{m}_p^{(0)} = \mathbf{0}$ as a start vector:

$$\Delta \kappa_p^{(1)} = \Delta \kappa_p^{*} + \Delta \kappa_p^{**D'}. \tag{B.5}$$

Inserting the components of (B.5) into (B.4) renders the first approximation of the strains $\Delta \epsilon_{prs}^{(1)}$, where $\Delta n_{pr}^{(0)} = 0$ is applied consequently.

In ideal elastic plastic materials there is $\Delta \epsilon_{prs}^{(N)} = \Delta \epsilon_{prs}$ in yielding parts of cross-section, and zero otherwise. Thus, a first approximation for the fictitious temperature loading may be calculated:

$$\Delta m_{pr}^{(1)} = \frac{1}{J} \sum_s z_s \Delta \epsilon_{prs}^{(N,1)} \Delta A_s, \tag{B.6}$$

$$\Delta n_{pr}^{(1)} = \frac{1}{A} \sum_{s} \Delta \epsilon_{prs}^{(N,1)} \Delta A_s. \tag{B.7}$$

Using eqn (B.6), a second approximation $\Delta \kappa_{pr}^{(2)}$ is calculated from (4.11); accordingly, $\Delta \epsilon_{prs}^{(2)}$ is derived from (B.4), where $\Delta n_{pr}^{(1)}$ from (B.7) is used. This, in turn, results in second approximations $\Delta m_{pr}^{(2)}$, $\Delta n_{pr}^{(2)}$ and so on, until the iteration is stopped by numerically limiting the changes.